

Projection Methods

Quantitative Macroeconomics

Raül Santaaulàlia-Llopis

MOVE-UAB and Barcelona GSE

Fall 2018

- In macroeconomics we are usually interested in solving for functional equations.
- Functional equations are difficult to solve because the unknown is not simply a vector in R^n , but an entire function. Usually, functional equations lack explicit closed-form solutions, hence, we look for approximate solutions that satisfy the functional equation closely.
- We can compute accurate approximate solutions to functional equations using techniques that are natural extensions of interpolation methods.

Projection Methods (aka Weighted Residual Methods)

- Projection methods solve for the unknown function f in

$$D(f) = 0$$

by specifying a linear combination:

$$\tilde{f}(x) = \sum_{j=1}^n \theta_j \psi_j(x)$$

with n -basis coefficients θ_j to be determined. The set of basis functions depends on the vector of state variables x .

Note that in the functional equation $D(f) = 0$ the operator D is known (e.g., an Euler Equation, a Bellman Equation, etc).

- For practical we choose the family of functions \mathcal{F} that collects functions that can be written as a linear combination of a set of n known linearly independent basis functions $\psi_j, j = 1, \dots, n$.

- Different choices of bases and of the projection algorithm will imply different projection methods. These alternative projections are often called in the literature by their own particular names, which can be sometimes bewildering.
- Projection theory, which has been applied in ad hoc ways by economists over the years, was popularized as a rigorous approach in economics by Judd (1992) and Gaspar and Judd (1997) and, as in the case of perturbation, it has been authoritatively presented by Judd (1998).

The Projection Algorithm

- 1 Define n known linearly independent functions $\psi_j : \Omega \rightarrow \mathfrak{R}$ where $n < \infty$. We call $\psi_1, \psi_2, \dots, \psi_n$ the basis functions. These basis functions depend on the vector of state variables x .
- 2 Define a vector of coefficients $\theta^l = [\theta_1^l, \dots, \theta_n^l]$ for $l = 1, \dots, m$ where m is the dimension that the function f of interest maps into. Stack all coefficients on a $m \times n$ matrix $\theta = [\theta^1, \dots, \theta^m]$.
- 3 Define a combination of the basis functions and the θ 's:

$$\tilde{f}^{l,n}(\cdot|\theta^l) = \sum_{j=1}^n \theta_j^l \psi_j(\cdot)$$

for $l = 1, \dots, m$. Then,

$$\tilde{f}^n(\cdot|\theta) = [\tilde{f}^{1,n}(\cdot|\theta^1); \tilde{f}^{2,n}(\cdot|\theta^2); \dots; \tilde{f}^{m,n}(\cdot|\theta^m)]$$

- 4 Plug $\tilde{f}^n(\cdot|\theta)$ into the operator $D(\cdot)$ to find the residual,

$$R(\cdot|\theta) = D(\tilde{f}^n(\cdot|\theta))$$

The Projection Algorithm (continued)

- 5 Find the value of θ that makes the residual equation close to 0 as possible given some objective function ρ ,

$$\theta = \arg \min_{\theta \in \mathbb{R}^{m \times n}} \rho(R(\cdot|\theta), 0)$$

Objective Function (Weighted Residuals)

- Define a residual equation as a functional equation evaluated at the approximation

$$R(x; \theta) = D(\tilde{f}(x; \theta))$$

- We want to choose θ to minimize $R(x; \theta)$ for all x given some metric. This step is known as the “projection” D against that basis to find the components of θ .

- We want to get the residual very close to zero in the weighted integral sense choosing θ so that

$$\int_X \omega(x)R(x; \theta)dx = 0$$

where $\omega(x)$ are the weight functions $\omega(x) = \sum_i \omega_i \phi_i(x)$ with ω_i being non-zero.

- Instead of setting $R(x; \theta) = 0$ for all $x \in X$, the weighted residual method sets a weighted integral of R to zero.
- We need to choose what $\phi_i(x)$ is. We consider three specific sets of weight functions:

- Weights are

$$\phi_i(x) = \frac{\partial R(x; \theta)}{\partial \theta_i}$$

- This set of weights is the set of first-order derivatives in:

$$\min_{\theta} \int_X R(x, \theta)^2 dx$$

- Weights are

$$\phi_i(x) = \delta(x - x_i)$$

where δ is the Dirac delta function.

- This set of weights implies that the residual is set to zero at n points x_1, \dots, x_n called the *collocation points*:

$$R(x_i; \theta) = 0 \quad i = 1, \dots, n$$

- If the set of basis functions (note that the weights ϕ_i are not necessarily equal to the basis ψ_i) is chosen from a set of orthogonal polynomials with collocation points given as the roots of the n th polynomial in the set, the method is called *orthogonal collocation*.

- Weights are

$$\phi_i(x) = \psi_i(x)$$

- The set of weight functions is the same as the basis functions used to represent \tilde{f} . That is, the Garlekin method forces the residual to be orthogonal to each of the basis functions.
- If the basis functions are chosen from a complete set of functions and given that enough terms are included, then the approximation \tilde{f} is the exact solution.

The steps in between

The projection algorithm require us to be familiar with methods to approximate functions, solve nonlinear systems, differentiate, integrate, and numerical optimization that we will see next in class.