Permanent Income Life Cycle Hypothesis Models

Growth and Development

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The Model

- Permanent income - Life Cycle Hypothesis (PILCH) models assume that agents do not have access to a complete set of contingent consumption claims.
  - ▷ Franco Modigliani and Albert Ando stressed the issues surrounding the finiteness of life and planning for retirement (Life-Cycle Hypothesis).
  - ▷ Friedman stressed issues surrounding income fluctuation (Permanent Income Hypothesis).[1]

- We see the main difference from complete markets looking at the Budget Constraint

$$c_t(s^t) + q_t(s^t)a_{t+1}(s^t) = y_t(s^t) + a_t(s^{t-1})$$

  where $a_{t+1}(s^t)$ is a one-period risk free IOU (I owe you) only, that is, no explicit insurance is allowed, only self-insurance via asset accumulation.

- Notice that all variables are functions of s^t but not s_{t+1} (same payoff in all states).

[1] These were hypothesis about behavior, the authors never wrote a model.

- The basic income fluctuation problem with exogenous incomplete markets is then

$$\max_{c(s^t), a_{t+1}(s^t)} \sum_{t=0}^{T} \sum_{s^t} \beta^t \pi_t(s^t) U^i(c^i(s^t), s^t) \tag{1}$$

subject to

▷ budget constraint

$$c_t(s^t) + q_t(s^t) a_{t+1}(s^t) = y_t(s^t) + a_t(s^{t-1}) \tag{2}$$

▷ a no Ponzi condition (assumed to be sufficient wide so as to never bind at the optimal allocation), normally a short-sale constraint on assets.

▷ an initial condition $a_0(s_{-1}) = a_{-1}$.

1. The income process of the individual is stochastic and given by $\{y_t\}_{t=0}^{T}$

2. The preference shock affecting individuals' preferences is a deterministic sequence $\{s_t\}$.

3. When $T = \infty$ we impose a short sale constraint on IOU's that prevents Ponzi scheme, but loose enough to allow optimal consumption smoothing, and notice that when $T < \infty$ we do not require no Ponzi scheme condition as long as we do not allow anybody to die in debt. The short sale constraint we impose is of the form, say

$$a_{t+1}(s^t) \geq -\sup_t \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{\tau-1}} \pi_\tau(s^\tau) \frac{y_\tau(s^\tau)}{(1+r)^{\tau-(t+1)}}$$

- The problem households solve is

$$\max \sum_{t=0}^{T} \sum_{s^t} \beta^t \pi_t(s^t) U^i(c^i(s^t), s^t)$$

s.t.

$$c_t(s^t) + q a_{t+1}(s^t) = y_t(s^t) + a_t(s^{t-1}).$$

- The price of a one period bond $q = \frac{1}{1+r}$ is nonstochastic and constant over time.

- $$\max \sum_{t=0}^{T} \sum_{s^t} \beta^t \pi_t(s^t) U^i(y_t(s^t) + a_t(s^{t-1}) - q a_{t+1}(s^t), s^t)$$

  Take first order conditions with respect to $a_{t+1}(s^t)$

  $$-q\beta^t \pi_t(s^t) U_c^i(c_t(s^t), s^t) + \beta^{t+1} \sum_{s^{t+1}|s^t} \pi_{t+1}(s^t) U_c^i(c_{t+1}(s^{t+1}), s^{t+1}) = 0 \quad (3)$$

  that is,

  $$U_c^i(c_t(s^t), s^t) = \frac{\beta}{q} \frac{\sum_{s^{t+1}} \pi_{t+1}(s^{t+1})}{\pi_t(s^t)}$$

- That is, for each of all given pair t and s^t we can also write as,

$$U_c^i(c_t(s^t), s^t) = \left(\frac{1+r}{1+\rho}\right) \sum_{s^{t+1}|s^t} \pi_{t+1}(s^{t+1}|s^t) \left[U_c^i(c_{t+1}(s^{t+1}), s^{t+1})|s^t\right] \quad (4)$$

or more compactly,

$$U_c^i(c_t, s^t) = \left(\frac{1+r}{1+\rho}\right) E_t \left[U_c^i(c_{t+1}, s^{t+1})\right] \quad \textit{for all given pair } t \textit{ and } s^t \quad (5)$$

where E_t is the expectation of s^{t+1} conditional on s^t.

- Equation (26) is the standard Euler Equation for PILCH type models.

Partial Equilibrium

- ▷ **interest rate** process is **exogenously given** (partial equilibrium)
- ▷ agents can **only self-insure against income fluctuations by** trading **1-period risk free bonds**[2]

- This implies relaxing 2 assumptions underlying the martingale hypothesis:
  - ▷ Incorporate a **precautionary saving motive** into the model (relax the assumption of linear marginal utility under uncertainty) so that agents reduce current consumption (increase savings) as reaction to an increase of uncertainty w.r.t. future labor income (Kimball 1990 and Deaton 1991)
  - ▷ Incorporate some **potentially binding borrowing constraint** (a liquidity constraint) that may prevent agents to borrow as much as desired to smooth consumption over time.

[2] by borrowing and lending at a risk free rate

Precautionary savings motive (Prudence)

- Recall that certainty equivalence explicitly rules out precautionary saving
- The consumption function under certainty equivalence states that only the conditional (on t information) first moment of the y_{t+s}'s matters for the consumption choice, but not the conditional variance of future labor income.

A simple model and a general result

- 2-period model: income is know in period 0, y_0, and in period 1 income is stochastic.[3]

- Denote Y_1 the random income in period 1,

$$Y_1 = \bar{y}_1 + \tilde{Y}_1$$

  where $E_0 Y_1 = \bar{y}_1$ (conditional expectation) and $\tilde{Y}_1$ is a random variable with $E_0 \tilde{Y}_1 = 0$.

- Denote the expected present discounted value of lifetime labor income (with, remember, $r = 0$)

$$w = y_0 + \bar{y}_1 \tag{6}$$

[3] Assume also $r = \rho = 0$—results go through being more flexible with r and ρ but algebra gets messy.

- Consumption at date 1 is then,

$$c_1 = w - c_0 + \tilde{Y}_1. \tag{7}$$

that is , c_1 is stochastic and varies with the realization of Y_1.

- The Euler equation (ignoring family size or other shocks to preferences) is

$$u_c(c_0) = E_0 u_c(w - c_0 + \tilde{Y}_1) \tag{8}$$

- We want to determine how consumption in period 0 (hence savings, $s = w - c_0$) varies with $\sigma_y^2 = Var(\tilde{Y}_1)$, the degree of uncertainty about labor income in period 1.

- For concreteness assume $\tilde{Y}_1$ can take 2 values, either ε with probability $1/2$ or $-\varepsilon$ with probability $1/2$ with $0 < \varepsilon < \bar{y}_1$ so that $\sigma_y^2 = \varepsilon^2$.

- Totally differentiating the Euler equation w.r.t. ε we get:

$$u_{cc}(c_0)\frac{d\ c_0}{d\ \varepsilon} = \frac{1}{2}u_{cc}(w - c_0 + \varepsilon)\left(-\frac{d\ c_0}{d\ \varepsilon} + 1\right) + \frac{1}{2}u_{cc}(w - c_0 + \varepsilon)\left(-\frac{d\ c_0}{d\ \varepsilon} - 1\right) \quad (9)$$

- Thus, how much consumption in period 0 varies with the degree of uncertainty about labor income in period 1, $\frac{d\ c_0}{d\ \varepsilon}$, is

$$\frac{d\ c_0}{d\ \varepsilon} = \frac{\frac{1}{2}\left(u_{cc}(w - c_0 + \varepsilon) - u_{cc}(w - c_0 - \varepsilon)\right)}{u_{cc}(c_0) + \frac{1}{2}\left(u_{cc}(w - c_0 + \varepsilon) + u_{cc}(w - c_0 - \varepsilon)\right)} \tag{10}$$

- The denominator is unambiguously negative (as u is strictly concave).

- The numerator is positive iff

$$u_{cc}(w - c_0 + \varepsilon) - u_{cc}(w - c_0 - \varepsilon) > 0, \tag{11}$$

that is, iff $u_{ccc}(c) > 0$ for arbitrary $\varepsilon > 0$

- Thus, a sufficient (and necessary) condition for precautionary savings (increases in s due to increases in the uncertainty of future labor income) is a strictly convex marginal utility (third derivative of utility function is positive).

- Remarks

  ▷ Kimball 1990 defines 'prudence' as the intensity of the precautionary savings *motive*: the "propensity to prepare and forearm oneself in the face of uncertainty".

  ▷ Note the term prudence defines a property of the utility function that generates precautionary savings, but the term prudence is not precautionary savings. We could generate precautionary savings behavior without a prudence motive as we shall see later.

  ▷ Note that prudence is not risk aversion. Prudence (precautionary saving *motive*) is controlled by the convexity of the marginal utility while risk aversion by the concavity of the utility function.

A parametric example for the general model

Consider a model with many periods:

- T periods.

- In general, we can say nothing about the consumption age-profile analytically. But we may get some qualitative results under some assumptions.

- Recall the Euler equation,

$$u_c(c_t, s^t) = \left(\frac{1+r}{1+\rho}\right) E_t u_c(c_{t+1}, s^{t+1}) \tag{12}$$

- Assuming separability and CRRA utility this becomes,

$$1 = \left(\frac{1+r}{1+\rho}\right) E_t \left[\frac{c_{t+1}}{c_t}\right] 6{-}\sigma \quad (13)$$

$$1 = E_t \left[e^{\ln\left((\frac{1+r}{1+\rho})(\frac{c_{t+1}}{c_t})^{-\sigma} \right)} \right] \quad (14)$$

$$1 = E_t \left[e^{\sigma \ln c_t + \ln(1+r) - \ln(1+\rho)} e^{-\sigma \ln c_{t+1}} \right] \quad (15)$$

$$1 = e^{\sigma \ln c_t + \ln(1+r) - \ln(1+\rho)} E_t[e^{-\sigma \ln c_{t+1}}] \quad (16)$$

- Now, assume $\ln c_{t+1}$ is normally distributed with mean $\mu = E_t \ln c_{t+1}$ and variance σ_c^2.[4]

[4] This requires appropriate assumptions on the underlying income process. For instance, for the 2-period model above this requires that the random variable $z = \ln(\kappa + \tilde{Y}_1)$ is normally distributed where $\kappa = w - c_0$ is a constant.

- Now, since

$$E_t e^{-\sigma \ln c_{t+1}} = \int_{-\infty}^{\infty} e^{-\sigma u} \frac{e^{-\frac{(u-\mu)^2}{2\sigma_c^2}}}{\sqrt{2\pi\sigma_c}} du = e^{\frac{1}{2}\sigma^2\sigma_c^2 - \sigma E_t \ln c_{t+1}} \tag{17}$$

where u is a dummy argument for $\ln(c_{t+1})$ in the integration.

- Then, after some algebra manipulation we can write the Euler equation as,

$$e^{-\sigma E_t \Delta \ln(c_{t+1}) + \ln(1+r) - \ln(1+\rho) + \frac{1}{2}\sigma^2\sigma_c^2} = 1 \tag{18}$$

- which, taking logs,

$$E_t \Delta \ln(c_{t+1}) = \frac{1}{\sigma}\left(\ln(1+r) - \ln(1+\rho)\right) + \frac{1}{2}\sigma^2\sigma_c^2 \tag{19}$$

that is, CRRA utility introduces precautionary savings *motive*.

- and note, for comparison, that without uncertainty the Euler equation is,

$$1 = \left(\frac{1+r}{1+\rho}\right)\left[\frac{c_{t+1}}{c_t}\right]^{-\sigma} \tag{20}$$

- That is,

$$\Delta \ln(c_{t+1}) = \frac{1}{\sigma}(\ln(1+r) - \ln(1+\rho)) \tag{21}$$

- **Remark 1.** Consumption does not obey certainty equivalence. CRRA utility introduces a precautionary savings *motive* that tilts the consumption profile upward in expectation—consumption growth is higher in expectation in the uncertainty case.

- **Remark 2.** The size of precautionary savings is determined by the parameter σ controlling prudence for the CRRA utility function.

- **Remark 3.** All variables v_t that at t help to predict the variability of future consumption σ_c^2 will help to predict expected consumption growth.

  Example: agents with higher income (or assets) since they may be better equipped to smooth consumption (future uncertainty) should have lower future consumption variability and thus, according to (14) lower consumption growth (see, for instance, Carroll (1992)). Thus, it may be flawed to run the regression,

$$\ln(c_{t+1}) = \alpha_1 + \alpha_2 \ln c_t + \alpha_3 v_t + \epsilon_t \tag{22}$$

  where they interpret a statistically significant estimate of $alpha_3$ as evidence against the PILCH model with CRRA utility.

Liquidity constraints

- So far, individuals can borrow up to the no-Ponzi scheme condition (arbitrary large amount) assumed generous enough never to be binding.

- However, borrowing constraints seem empirically plausible and formal econometric tests indicate so (see Zeldes 1989).

The Euler equation with liquidity constraints

- Let's assume there exists borrowing constraints, for simplicity (following Aiyagari (1994)) assume $a_{t+1}(s^t) \geq 0$—that is, agents cannot borrow at all.
- These constraints may or may not be binding, depending on the realizations of the labor income shock, but we have to take these constraints into account explicitly when deriving the stochastic Euler equation.
- Attach the Lagrange multiplier $\mu(s^t)$ to the borrowing constraint $a_{t+1}(s^t) \geq 0$ at event history s^t.

- FOC w.r.t. consumption today and tomorrow remain unchanged:

$$\beta^t \pi_t(s^t) u_c(c_t(s^t), s^t) = \lambda(s^t) \tag{23}$$

$$\beta^{t+1} \pi_{t+1}(s^{t+1}) u_c(c_{t+1}(s^{t+1}, s^{t+1}) = \lambda(s^{t+1}) \tag{24}$$

- FOC w.r.t. $a_{t+1}(s^{t+1})$ becomes,

$$\frac{\lambda(s^t)}{1+r} - \mu(s^t) = \sum_{s^{t+1}|s^t} \lambda(s^{t+1}) \tag{25}$$

with complementary slackness conditions,

$$a_{t+1}(s^t)\mu(s^t) \geq 0 \tag{26}$$

$$a_{t+1}(s^t)\mu(s^t) = 0 \tag{27}$$

- Combining the FOC yields,

$$u_c(C_t(s^t)) - \mu(s^t)(1+r) = \beta(1+r) \sum_{s^{t+1}|s^t} \beta\pi_{t+1}(s^{t+1}|s^t) u_c(c_{t+1}(s^{t+1})) \tag{28}$$

- or in short,

$$\begin{aligned} u_c(c_t) &\geq \frac{1+r}{1+\rho} E_t u_c(c_{t+1}) & (29) \\ &= \frac{1+r}{1+\rho} E_t u_c(c_{t+1}) \quad \textit{if} \quad a_{t+1} > 0 & (30) \end{aligned}$$

- This Euler equation sometimes holds with inequality, depending on whether the liquidity constraint is binding.

- Note also from the budget constraint that

$$\begin{aligned} c_t &= y_t + a_t - \frac{a_{t+1}}{1+r} && (31) \\ &\leq y_t + a_t && (32) \end{aligned}$$

because of the constraint $a_{t+1} \geq 0$. Thus,

  - ▷ either $a_{t+1} = 0$, therefore $c_t = y_t + a_t$, and

$$u_c(c_t) = u_c(y_t + a_t) \quad (33)$$

  - ▷ or $a_{t+1} > 0$, therefore $c_t < a_t + y_t$ and thus (using strict concavity of the utility function)

$$u_c(y_t + a_t) < u_c(c_t) = \frac{1+r}{1+\rho} E_t u_c(c_{t+1}) \quad (34)$$

- Hence, **we can write the Euler equation compactly** as

$$u_c(c_t) = \max\left\{u_c(y_t + a_t), \frac{1+r}{1+\rho}E_t u_c(c_{t+1})\right\} \tag{35}$$

Liquidity constraints with quadratic preferences (No Prudence)

- With quadratic preferences the Euler Equation becomes,

$$-(c_t - \bar{c}) = \max\left\{-(y_t + a_t - c_t), -\frac{1+r}{1+\rho}(E_t c_{t+1} - \bar{c})\right\} \tag{36}$$

- For simplicity assume $r = \rho$, then we can write the Euler equation as,

$$\begin{aligned} c_t &= \min\{y_t + a_t, E_t c_{t+1}\} & (37) \\ &= \min\{y_t + a_t, E_t \min\{y_{t+1} + a_{t+1}, E_{t+1} c_{t+2}\}\} & (38) \end{aligned}$$

and so forth.

An important implication

- Suppose the variance of future income increases (say for $t + 1$), making more low realizations of y_{t+1} possible (and more high realizations too).

- If the set of y_{t+1} values for which the borrowing constraint binds becomes bigger, $E_t \min\{y_{t+1} + a_{t+1}, E_t c_{t+2}\}$ declines and so does c_t, since in more instances the min is the first of the 2 objects.

- That is, saving increases in reaction to increases in uncertainty about future income, because agents, afraid of future contingencies of low consumption smooth low income shocks via borrowing, increase their precautionary savings.

- **The presence of liquidity constraints (+ risk aversion) generates precautionary savings behavior without the need of a prudence motive.**

- Hence, when we observe increases in savings as a reaction to increased income uncertainty, this may have a preference-based interpretation (agents are prudent: $u_{ccc} > 0$ or an institution-based interpretation (credit markets prevent or limit uncontingent borrowing).

A second important implication

- In the absence of borrowing constraints we know from our previous discussion that

$$c_t = E_t(c_{t+s}) \tag{39}$$

  for all $s > 0$. Suppose there exists an $s > 0$ (say, $s = 1$) such that for some realization of the income shock y_{t+s} (with positive probability) the agent is borrowing constrained, and thus $y_{t+s} + a_{t+s} < E_t c_{t+s}$.

- Then, using the law of iterated expectations, $c_t = \min\{y_t + a_t, E_t c_{t+1}\} < E_t c_{t+s}$. Thus, even if the liquidity constraint is not binding in period t, future potential borrowing constraints affect current consumption choices.

Numerical solutions of models with precautionary saving and liquidity constraints

- For computational purposes we want to make the problem a consumer faces recursive.
- The stochastic process governing labor income is described by a finite state, stationary Markov process with domain $y \in Y = \{y_1, \ldots, y_N\}$ and transition probabilities $\pi(y'|y)$ where we assume $y_1 \geq 0$ and $y_{i+1} > y_i$.
- As state variables we choose current asset holdings and the current labor income shock (a, y).

Borrowing limits

- In order to make the problem well-behaved we have to make sure that agents do not go into debt so much that they cannot pay at least the interest on that debt and still have non-negative consumption.
- Let $\bar{A}$ be the max amount an agent is allowed to borrow.
- Since consumption equals $c = y + a - \frac{a'}{1+r}$, we have, for an agent that *i)* borrowed the max amount $a = -\bar{A}$, *ii)* received the worst income shock y_1 and *iii)* just repays interest (i.e. $a' = \bar{A}$):

$$c = y_1 + \bar{A} - \frac{\bar{A}}{1+r} \geq 0 \tag{40}$$

Non-negativity of consumption implies the borrowing limit,

$$\bar{A} = \frac{1+r}{r} y_1 \tag{41}$$

which we impose on the consumer.

- Since $\bar{A}$ is also the present discounted value of future labor income in the worst possible scenario of always obtaining the lowest income realization y_1, we may call this borrowing limit the *natural debt limit* (Aiyagari 1994).
- Note that, since borrowing up to the borrowing limit implies a positive probability of zero consumption next period (if $\pi(y_1|y) > 0$ for all $y \in Y$), this borrowing constraint is not going to be binding, as long as the utility function satisfies the Inada conditions.

Bellman's equation

- For any $a \in [\bar{A}, \infty]$ and any $y \in Y$ we then can write Bellman's equation as,

$$v_t(a, y) = \max_{-\bar{A} \leq a' \leq (1+r)(a+y)} u\left(y + a - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y'|y) v_{t+1}(a', y') \tag{42}$$

- The first order condition to this problem is,

$$\frac{1}{1+r} u'\left(y + a - \frac{a'}{1+r}\right) = \beta \sum_{y'} \pi(y'|y) v'_{t+1}(a', y') \tag{43}$$

where v'_{t+1} is the derivative of v_{t+1} w.r.t. its first argument.

Euler equation

- The envelope condition is,

$$v_t'(a, y) = u'\left(y + a - \frac{a'}{1+r}\right) \tag{44}$$

which we can forward one period.

- Combining FOC and envelope yields,

$$u'\left(y + a - \frac{a'}{1+r}\right) = \beta(1+r)\sum_{y'} \pi(y'|y)u'\left(y' + a' - \frac{a^{''}}{1+r}\right) \tag{45}$$

- Defining $c_t = y + a - \frac{a'}{1+r}$ and $c_{t+1} = y' + a' - \frac{a^{''}}{1+r}$ we are back to our previously derived stochastic Euler equation.

Computation of partial equilibrium PILCH models

- As we have seen in class previously, we can either work on the value function directly or work on the Euler equations to solve for optimal policies.
- Also, we have seen that depending on the time horizon of the agent (finite versus infinite), different iterative procedures can be applied.

(a) Infinite $T = \infty$, value function iteration

- We want to find a time-invariant value function $v(a, y)$ and associated policy functions $c(a, y)$, $a'(a, y)$.

- We need to find a fixed point to Bellman's equation, since there is no final period to start from.

- Thus, we make an initial guess for the value function $v^0(a, y)$ and then iterate on the functional equation,

$$v^n(a, y) = \max_{-\bar{A} \leq a' \leq (1+r)(a+y)} u\left(y + a - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y'|y) v^{n-1}(a', y') \qquad (46)$$

until it converges,

$$\|v^n - v^{n-1}\| \leq \varepsilon_v$$

- Note that at each iteration we generate policy functions ${a'}^n(a, y)$ and $c^n(a, y)$.

- We know that under appropriate assumptions ($\beta < 1$ and u bounded), the operator defined by Bellman's equation is a contraction mapping and hence convergence of the iterative procedure to a unique fixed point is guaranteed.

(b) Finite T, value function iteration

- We want to find sequences of value functions $\{v_t(a, y)\}_{t=0}^{T}$ and associated policy functions $\{c_t(a, y), a_{t+1}(a, y)\}_{t+0}^{T}$.
- Given that agent dies at period T we can normalize $v_{T+1} \equiv 0$
- Then we can iterate backwards on the equation (41): at period t we know the function $v_{t+1}(\cdot, \cdot)$, hence we can solve the maximization problem to find functions $a'_t(\cdot, \cdot), v_t(\cdot, \cdot)$ and $c_t(\cdot, \cdot)$.
- Note that for each t there are as many maximization problems to solve as there are admissible (a, y)-pairs.

(c) Infinite $T = \infty$, policy function iteration on the Euler equation

- We have no initial period, so we guess an initial policy $a'^0(a, y)$ or $c^0(a, y)$ and then iterate on

$$u'\left(y + a - \frac{{a'_t}^n(a, y)}{1 + r}\right) = \beta(1 + r) \sum_{y'} \pi(y'|y) u'\left(y' + {a'_t}^n - \frac{{a'_{t+1}}^{n-1}({a'_t}^n(a, y), y')}{1 + r}\right)$$

  until it converges,

$$\|{a'}^n - {a'}^{n-1}\| \leq \varepsilon_a$$

- Deaton (1992) argues that under the assumption $\beta(1 + r) < 1$ the operator defined by the Euler equation is a contraction mapping, so that convergence to a unique fixed point is guaranteed.

(d) Finite T, policy function iteration on the Euler equation

- We want to find sequences of policy functions $\{c_t(a,y), a'_t(a,y)\}_{t=0}^T$. Again, given that the agent dies at period $T+1$ we know that at period T all income will be consumed and nothing saved, thus

$$c_T(a,y) = a + y$$
$$a'_T(a,y)$$

- The Euler equation between period t and $t+1$ reads as

$$u'\left(y + a - \frac{a'_t(a,y)}{1+r}\right) = \beta(1+r)\sum_{y'}\pi(y'|y)u'\left(y' + a'_t(a,y) - \frac{a'_{t+1}(a'_t(a,y),y')}{1+r}\right)$$

  where $a'_{t+1}(a'_t(a,y),y')$ is a known function from the previous step and we want to solve for the function $a'_t(\cdot,\cdot)$.

- Note that for a given (a,y) (contrary to the previous procedure, VFI) no maximization is needed to find $a'_t(a,y)$, one just has to find a solution to a (potentially higher nonlinear) single equation. And we have learned how to do that in the first half of this course.

Simulations of time paths

- Once one has solved for the policy functions one can simulate time paths (of say M periods) for consumption and asset holdings.

- Start at some asset level a_0, possibly equal to zero, then draw a sequence of random income numbers $\{y_i\}_{t=0}^{M}$ according to the specified income process.

- Now one can generate a time series for consumption and asset holdings for the infinite horizon case by

$$\begin{aligned} c_0 &= c(a_0, y_0) \\ a_1 &= a'(a_0, y_0) \end{aligned}$$

  and recursively,

$$\begin{aligned} c_i &= c(a_i, y_i) \\ a_{i+1} &= a'(a_i, y_i) \end{aligned}$$

- The same applies for the finite horizon case, but here the policy functions also vary with time.

What if the income process follows a random walk?

- Similar treatment to how we deal with RA models with stochastic trends.
- See Deaton's example in Krüger's notes.

Alternative Solution Method: Endogenous grid

- Chris Carroll (2008) idea: Construct a grid on a', next period's asset holdings, rather than on a —as standardly done.

- Necessary condition: policy function must be at least weakly monotonic in current assets.

A model

- Recursively, agents solve

$$V(a, y) = \max_{a'} u(c) + \beta E_t V(a', y')$$

subject to

$$\begin{aligned} c + a' &\leq y + (1 + r)a \\ a' &\geq a_{min} \end{aligned}$$

- With a discrete income process we can write the Euler equation as,

$$u_c(c(a,y)) \geq \beta(1+r) \sum_{y' \in Y} \pi(y'|y) u_c(c(a',y'))$$

subject to

$$\begin{aligned} c + a' &\leq y + (1+r)a \\ a' &\geq a_{\min} \end{aligned}$$

- Equality holds if $a' > a_{\min}$

- We want to find an invariant decision rule $c(a, y)$ (hence $a'(a, y)$) that satisfies the Euler equation and that does not violate the borrowing constraint.

- We will use Euler equation methods, iterating on the decision rule. For mere exposition purposes, think of a descretization of the decision rule but continuous methods apply as well.

The algorithm

- **Step 1.** Construct a grid on (a', y) where $a' \in G_A = \{a_1, \ldots, a_{max}\}$ with $a_1 = a_{\min}$ and $y \in G_y = \{y_1, \ldots, y_N\}$. Important: we are defining our grid over assets tomorrow.

- **Step 2.** Guess a decision rule $c_0(a_i, y_j)$.

  For example, guess $a'_0(a, y) = 0$, that is, $c_0(a_i, y_j) = (1 + r)a_i + y_j$.

- **Step 3.** For any pair $\{a_i', y_j\}$ on the mesh $G_A \times G_y$ construct the RHS of the Euler equation,

$$RHS(a_i', y_j) = \beta(1+r) \sum_{y' \in Y} \pi(y'|y_j) u_c(c_0(a_i', y'))$$

- **Step 4.** Use the Euler equation to solve for the value $\tilde{c}(a_i', y_j)$ that satisfies

$$u_c(\tilde{c}(a_i', y_j)) = RHS(a_i', y_j)$$

which can be done analytically. For example, for $u_c(c) = c^{-\gamma}$ we have:

$$\tilde{c}(a_i', y_j) = \left[RHS(a_i', y_j)\right]^{-\frac{1}{\gamma}} .$$

This is the step that makes the algorithm more efficient:

  - ▷ We do not require a nonlinear equation solver, and
  - ▷ We only compute the expectation in **Step 3** once.

- **Step 5.** From the budget constraint, solve for $\tilde{a}(a'_i, y_j)$ such that

$$\tilde{c}(a'_i, y_j) + a'_i = (1+r)\tilde{a}(a'_i, y_j) + y_j$$

  - ▷ Where $\tilde{a}(a'_i, y_j)$ is the value of assets today that would lead the consumer to have a'_i assets tomorrow if this income shock was y_j today.
  - ▷ Note this function is not necessarily defined on the grid points of G_A.
  - ▷ This is the endogenous grid and it changes on each iteration.
  - ▷ This way, $\tilde{a}(a'_{min}, y_j)$ is the value for current assets that induces the borrowing constraint to bind next period, i.e., $a'_i = a_{min}$.

- **Step 6.** Update the guess. To get the new guess $c_1(a_i, y_j)$:

  For each y_j,

  - If $a_i > \tilde{a}(a'_{min}, y_j)$, use some interpolation methods on the two most adjacent values $\{\tilde{a}_n(a_i, y_j), \tilde{a}_{n+1}(a_i, y_j)\}$ that include the given grid point a_i.

  - If $a_i < \tilde{a}(a'_{min}, y_j)$, use the budget constraint

    $$c_1(a_i, y_j) = (1 + r)a_i + y_j - a'_{min}$$

    since we cannot use the Euler equation because the borrowing constraint is binding.

- **Step 7.** Check convergence and stop when

$$\max_{i,j}\{|c_{n+1}(a'_i, y_j) - c_n(a'_i, y_j)|\} < \epsilon$$

If convergence is not achieve, go back to **Step 3.**

General Equillibrium

- Now we are ready for a fully-fledged Macroeconomic Model with Heterogeneous agents.
- This is the General Equilibrium version of PILCH models introduced in previous sessions.
- General Equilibrium implies that aggregate prices w_t and r_t are determined endogenously within the model.

  ▷ It imposes theoretical discipline. The relationship between ρ and r will obey an equilibrium —rather than being arbitrarily chosen.

  ▷ It generates an endogenously determined consumption and wealth distribution and hence provides a theory of wealth inequality —that is, it provides a theoretical framework potentially able to account for stylized facts of empirical wealth distributions.

  ▷ It enables meaningful policy experiments. Partial equilibrium models (small open economies) with $\bar{w}$ and $\bar{r}$ may over- or underestimate the full effects of reforms.

A Macroeconomic Model with Heterogeneous Agents

- The economy is populated by a continuum of measure 1 of individuals that all face an income fluctuation problem as described previously.
- Each individual is subject to the same stochastic labor endowment process $\{y_t\}$ with $y_t \in Y = \{y_1, y_2, \ldots, y_N\}$ —each individual will face potentially different realizations of this process.
- Individual labor income at t is $w_t y_t$ where w_t is the real wage per unit of labor. Hence, y_t, can be interpreted as the number of efficiency units of labor a household can supply in a given time period.

- The **labor endowment process is assumed to follow a stationary Markov process**. Let $\pi(y'|y)$ denote the probability that tomorrow's endowment takes the value y' if today's endowment takes the value y.

- Not only $\pi(y'|y)$ is the probability of a particular agent of a transition from y to y' but also, by the **law of large numbers**[5] the deterministic fraction of the population that has this particular transition.

- Let Π be the stationary distribution associated with π, assumed to be unique.

[5] See Uhlig (1996)

- Assume that at $t = 0$ the income of all agents, y_0, is given, and that the distribution of incomes across the population is given by Π. Then the distribution of income in all future periods is also given by Π, in particular, the total labor endowment in the economy (in efficiency units) is given by:

$$\bar{L} = \sum_{y} y\Pi(y)$$

- Hence, although there is substantial idiosyncratic uncertainty about a particular individual's labor endowment and hence labor income is constant over time, that is, there is no aggregate uncertainty.

- Denote $\pi_t(y^t|y_0)$ denote the probability of event history y^t, given initial event y_0. Then,

$$\pi_t(y^t|y_0) = \pi(y_t|y_{t-1})\pi(y_{t-1}|y_{t-2})\dots\pi(y_1|y_0)$$

- Each agent's preferences over stochastic consumption processes are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

with $\beta = \frac{1}{1+\rho}$ and $\rho > 0$.

- The agent can self-insure against idiosyncratic labor endowment shocks by purchasing at period t uncontingent claims to consumption at period $t+1$. Then, agent's budget constraint is[6]

$$c_t + a_{t+1} = w_t y + t + (1 + r_t) a_t$$

- We impose an exogenous borrowing constraint on asset holdings: $a_{t+1} > 0$, that is, a no-borrowing constraint, as in the Aiyagari's (1994) applications.

- Given (a_0, y_0).

[6] Note that instead of zero coupon bonds being traded at a discount $q = \frac{1}{1+r}$ we now consider a bond that trades at price 1 today and earns gross interest rate $(1 + r_{t+1})$ tomorrow. Three remarks: the latter option seems more suitable now because the asset being traded will be physical capital with the interest rate being determined by the marginal product of capital: as long as the interest rate is constant as in Aiyagari (1994) both formulations are equivalent (identical Euler equations): and, with aggregate uncertainty as in Krusell and Smith (1998), however, it would make a substantial difference whether agents can trade a risk free bond or, as Krusell and Smith assume, risky capital.

- Some notation,

  - ▷ Let $c_t(a_0, y^t)$ and $a_{t+1}(a_0, y^t)$ denote, respectively, consumption and asset holdings at period t after endowment shock history y^t is realized.
  - ▷ Let $\Phi_0(a_0, y_0)$ denote the initial measure over (a_0, y_0) across households.
  - ▷ Let the marginal distribution of Φ_0 with respect to y_0 be Π.
  - ▷ At each point in time each agent is characterized by her current asset position a_t and her current income y_t. Those are agent's individual state variables.
  - ▷ The aggregate state of the economy is the cross-sectional distribution over individual characteristics $\Phi_t(a_t, y_t)$.

- On the production side:

  ▷ Assume competitive market taken factor prices as given. All firms have access to a standard neoclassical production technology,

  $$Y_t = F(K_t, L_t)$$

  where $F \in \mathcal{C}^2$ has CRS and positive but diminishing marginal products with respect to both production factors. Inada conditions also hold.

  ▷ Firms choose labor inputs and capital inputs to maximize the present discounted value of profits.

  ▷ Remember that with CRS and perfect competition the number of firms is indeterminate and w/o loss of generality we can assert the existence of a single representative firm.

- The only asset in this economy is the physical capital stock, hence, in equilibrium the aggregate capital stock K_t has to equal the sum of asset holdings of all individuals, that is, the integral over the a_t's of all agents.

- Capital depreciates at rate $0 < \delta < 1$.

- The aggregate resource constraint is then

$$C_t + K_{t+1} = F(K_t, L_t) + (1 - \delta)K_t$$

where C_t is aggregate consumption at t.

- We are now ready for the definition of an equilibrium. We will look at three definitions:
  - ▷ Sequential markets equilibrium: to compare this model to the complete markets one.
  - ▷ Stationary recursive equilibrium: what we will learn to compute numerically.
  - ▷ Recursive equilibrium: that generalizes the stationary one, and it is what is needed when we introduce aggregate uncertainty.

Definition (Sequential markets equilibrium, SME)

Given Φ_0, a sequential markets competitive equilibrium is allocations for households $\{c_t(a_0, y^t), a_{t+1}(a_0, y^t)\}_{t=0,y^t\in Y}^{\infty}$, allocations for the representative firm $\{K_t, L_t\}_{t=0}^{\infty}$ and prices $\{w_t, r_t\}_{t=0}^{\infty}$ such that,

1. Given prices, allocations maximize agent's problem subject to agent's budget constraint, the nonnegativity constraints on assets and consumption and the borrowing constraint.
2. Firms maximize their profits (factor prices equate marginal productivities),

$$r_t = F_K(K_t, L_t) - \delta$$
$$w_t = F_L(K_t, L_t)$$

3. Forall t, markets clear, respectively assets, labor and goods:

$$K_{t+1} = \int \sum_{y' \in Y} a_{t+1}(a_0, y^t)\pi(y^t|y_0)d\Phi_0(a_0, y_0)$$
$$L_t = \bar{L} = \int \sum_{y' \in Y} Y_T \pi(y^t|y_0)d\Phi_0(a_0, y_0)$$
$$\int \sum_{y' \in Y} c_t(a_0, y^t)\pi(y^t|y_0)d\Phi_0(a_0, y_0) + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$$

- Let's define the **recursive competitive equilibrium**.

- First, we need to define a measurable space on which the measure Φ are defined.
    - ▷ Define the set $A = [0, \infty)$ of possible asset holdings and the set Y of possible labor endowment realizations.
    - ▷ Define by $\mathcal{P}(Y)$ the power set of Y (i.e. the set of all subsets of Y) and by $\mathcal{B}(A)$ the Borel σ-algebra of A.
    - ▷ Let $Z = A \times Y$ and $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(A)$.
    - ▷ Define $\mathcal{M}$ the set of all probability measures on the measurable space $M = (Z, \mathcal{B}(Z))$.
    - ▷ Why all this? Because measures Φ will be required to be elements of $\mathcal{M}$.

Household problem in recursive formulation

- In recursive formulation,

$$v(a, y; \Phi) = \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) v(a', y'; \Phi')$$

$$\begin{aligned} s.t. \quad c + a' &= w(\Phi)y + (1 + r(\Phi))a \\ \Phi' &= H(\Phi) \end{aligned}$$

where the function $H : \mathcal{M} \to \mathcal{M}$ is called the aggregate " law of motion".

- Note we must include all state variables in the household problem, in particular the aggregate state variable, since the interest rate r will depend on Φ.

Definition (Recursive competitive equilibrium, RCE)

A RCE is a value function $v : Z \times \mathcal{M} \rightarrow \mathcal{R}$, policy functions for the agent $a' : Z \times \mathcal{M} \rightarrow \mathcal{R}$ and $c : Z \times \mathcal{M} \rightarrow \mathcal{R}$, policy functions for the firm $K : \mathcal{M} \rightarrow \mathcal{R}$ and $L : \mathcal{M} \rightarrow \mathcal{R}$, pricing functions $w : \mathcal{M} \rightarrow \mathcal{R}$ and $r : \mathcal{M} \rightarrow \mathcal{R}$ and an aggregate law of motion $H : \mathcal{M} \rightarrow \mathcal{M}$ such that

1. Given w and r, v, a' and c are measurable with respect to $\mathcal{B}(Z)$, v satisfies the household's Bellman equation and a' and c are the associated policy functions
2. K, L satisfy, given w and r,

$$r(\Phi) = F_K(K(\Phi), L(\Phi)) - \delta \quad \textit{and} \quad w(\Phi) = F_L(K(\Phi), L(\Phi))$$

3. Markets clear. For all $\Phi \in \mathcal{M}$,

$$\begin{aligned} K(H(\Phi)) &= \int a'(a, y; \Phi)\, d\Phi \\ L(\Phi) &= \int y\, d\Phi \\ \int c(a, y; \Phi) d\Phi + \int a'(a, y; \Phi) d\Phi &= F(K(\Phi), L(\Phi)) + (1 - \delta) K(\Phi) \end{aligned}$$

4. The aggregate law of motion H is generated by the exogenous Markov process π and the policy fucntion a' (as described below).

- What does it mean that H is generated by π and a'?

  - ▷ H basically tells us how a current measure over (a, y) translates into a measure Φ' tomorrow.

  - ▷ So H summarizes how individuals move within the distribution over assets and income from one period to the next.

  - ▷ But that is exactly what a transition function tells us. Define the transition function $Q_\Phi : Z \times \mathcal{B}(Z) \rightarrow [0, 1p]$ by[7]

$$Q_\Phi((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in Y} \pi(y'|y) \ \textit{if} \ a'(a, y; \Phi) \in \mathcal{A}, \quad 0 \ \textit{else}$$

    for all $(a, y) \in Z$ and all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$.

  - ▷ $Q_\Phi(\mathcal{A}, \mathcal{Y})$ is the probability that an agent with current assets a and current income y ends up with assets a' in $\mathcal{A}$ tomorrow and income y' in $\mathcal{Y}$ tomorrow.

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[7] Since a' is a function of Φ, Q is also implicitly a function of Φ.

- How does the function Q_Φ help us to determine tomorrow's measure over (a, y) from today's measure?
  - ▷ Suppose Q_Φ were a Markov transition matrix for a finite state Markov chain and Φ_t would be the distribution today.
  - ▷ Then, to figure out the distribution Φ_{t+1} tomorrow we would just multiply Q by Φ_t,

    $$\Phi_{t+1} = Q_\Phi^T \Phi_t$$

    where T is the transposition operator.
  - ▷ But a transition function is just a generalization of a Markov transition matrix to uncountable state spaces. Hence, we need integrals:

    $$\Phi'(\mathcal{A}, \mathcal{Y}) = H(\Phi)(\mathcal{A}, \mathcal{Y}) = \int Q_\Phi((a, y), (\mathcal{A}, \mathcal{Y}))\ \Phi(da \times dy)$$

  - ▷ The fraction of people with income in $\mathcal{Y}$ and assets in $\mathcal{A}$ is the fraction of people today, as measured by Φ, that transit to $(\mathcal{A}, \mathcal{Y})$, as measured by Q_Φ.

- In general, there is no presumption that tomorrow's measure Φ' equals today's measure, since we posed an arbitrary initial distribution over types, Φ_0.

- If the sequence of measures $\{\Phi_t\}$ generated by Φ_0 and H is not constant, then:
  - ▷ obviously interest rates $r = r(\Phi_t)$ are not constant,
  - ▷ decision rules vary with Φ_t over time.

- That is, the computation of equilibria is difficult in general. We would have to compute the function H explicitly, mapping measures (i.e. infinite-dimensional objects) into measures. We will discuss that next week. Today we will be interested in stationary versions where Φ is constant.

Definition (Stationary recursive competitive equilibrium)

A stationary RCE is a value function $v : Z \to \mathcal{R}$, policy functions for the agent $a' : Z \to \mathcal{R}$ and $c : Z \to \mathcal{R}$, policies for the firm K and L, prices w and r and a measure $\Phi \in \mathcal{M}$ such that

1. Given w and r, v, a' and c are measurable with respect to $\mathcal{B}(Z)$, v satisfies the household's Bellman equation and a' and c are the associated policy functions.
2. K, L satisfy, given w and r,

$$r = F_K(K, L) - \delta \quad \textit{and} \quad w = F_L(K, L)$$

3. Markets clear,

$$\begin{aligned} K &= \int a'(a, y)\ d\Phi \\ L &= \int y\ d\Phi \\ \int c(a, y) d\Phi + \int a'(a, y)\ d\Phi &= F(K, L) + (1 - \delta)K \end{aligned}$$

4. For all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$,

$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y}))\ d\Phi$$

- Note the simplification with respect to the previous nonstationary problem: value functions, policy functions and prices are not any longer indexed by measures Φ.

- The last requirement states that the measure Φ reproduces itself: starting with measure over incomes and assets Φ today, generates the same measure tomorrow.

- In this sense, the stationary RCE is the equivalent of a steady state, only that the entity characterizing the steady state is not a number but a complicated infinite-dimensional object, a measure.

Theoretical results: existence and uniqueness

- We deal first with existence: it all boils down to whether one equation in one unknown, the interest rate, has a solution.

- By Walras law, we do not need the goods market. The labor market clearing is exogenously given (and inelastic supply) and specified by the labor endowment process so its equilibrium is easy.

- What remains is the asset market clearing:

$$K = K(r) = \int a'(a, y)\ d\Phi \equiv Ea(r)$$

where $Ea(r)$ are the average asset holdings in the economy. This condition requires equality between the demand for capital by firms and the supply of capital by households (last period's demand for assets, with physical capital being the only asset in the economy).

- It is clear that capital demand $K(r)$ is a function of r alone,

$$r = F_K(K(r), \bar{L}) - \delta$$

since labor supply $L = \bar{L} > 0$ is exogenous.

- Further, we know from the assumptions on the production function that $K(r)$ is a continuous, strictly decreasing function on $r \in (-\delta, \infty)$ with

$$\begin{aligned} \lim_{r \to -\delta} K(r) &= \infty \\ \lim_{r \to \infty} K(r) &= 0 \end{aligned}$$

- It is clear that $Ea(r)$ satisfies $Ea(r) \in [0, \infty)$ for all $r \in (-\delta, \infty)$.

- It is our goal to characterize $Ea(r)$, in particular, if $Ea(r)$ is continuous and satisfies

$$\lim_{r \to -\delta} Ea(r) < \infty$$
$$\lim_{r \to \infty} Ea(r) > 0$$

  then, a stationary RCE exists.

- Further, if $Ea(r)$ is strictly increasing in r (the substitution effect outweighs the income effect), then the stationary RCE is unique.

Computation of the general equilibrium

- Use your partial equilibrium programs from before (with liquidity constraints) to compute the general equilibrium for the aggregate economy.

- The algorithm goes like this:

Computational Algorithm for General Equilibrium

1. Guess an interest rate $r \in (\delta, \rho)$.

2. Use the first order conditions for the firm to determine $K(r)$ and $w(r)$.

3. Solve the household problem for given r and $w(r)$: Here you will use code of the previous exercises.

4. Use the optimal decision rule $a'(a, y)$ together with the exogenous Markov chain π to find an invariant distribution Φ_t associated with $a'(a, y)$ and π. For this you better first check that $a'(a, y)$ intersects the 45-degree line for a large enough. Note that if you have discretized the state space for assets, finding Φ_r amounts to finding the eigenvector (normalized to length one) associated with the largest eigenvalue of the transition matrix Q generated by $a'(a, y)$ and π.

5. Compute
$$Ea(r) = \int a'(a, y) d\Phi_r$$
Again, if you have discretized the state space, the integral really is a sum.

6. Compute
$$d(r) = K(r) - Ea(r)$$
If $d(r)$ is close enough to zero, you have found a stationary recursive equilibrium, if not, update your guess for r going back to (1).