Linearized Euler Equation Methods

Quantitative Macroeconomics [Econ 5725]

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Introduction

- Dynamic stochastic general equilibrium (DSGE) models are often characterized by a set of nonlinear equations, some of which are intertemporal Euler equations.[1]
- In previous sessions we have seen how to solve nonlinear systems.
- Here, we will use local approximations, and transform the nonlinear system posed by DSGE models into a system of linear equations. Now, some of these linearized equations are difference equations.
- A variety of methods can be used to solve for these DSGE models. First, we will focus on the Blanchard and Kahn (1980) method that relies on matrix decomposition.[2] Second, we go over the method of undetermined coefficients in Uhlig (1997) that applies even when certain matrix operations are not available.

[1] These slides borrow from notes by Larry Christiano, Jesús Fernández-Villaverde, Juan Rubio, José-Víctor Ríos-Rull, Makoto Nakajima, and Fabrizio Perri. See further references in the syllabus.

[2] Klein (2000) and King and Watson (2001) proposed more generalized matrix decomposition methods that resolves some noninvertibility issues. You are very encouraged to read their manuscripts.

- The methods based on linearized Euler equations are useful to solve more than pareto optimal economies. These include complete market economies with distortions (e.g. labor income taxation). That is, we can solve problems for which not necessarily a social planner (first best) solution exists.

- In future sessions, we will use global methods to solve for the decision rules using the Euler equations that we will not linearize. Our working examples will be models with market incompleteness, without and with aggregate risk.

The neoclassical growth model

- A continuum N_t of identical households that life infinitely maximize:

$$\max_{c_t \geq 0, k_{t+1} \geq 0, h_t \in [0,1]} \sum_{t=0}^{\infty} \beta^t N_t \; u(c_t, h_t)$$

subject to the budget constraint,

$$C_t + K_{t+1} = w_t \; H_t + (1 + r_t - \delta) \; K_t$$

where $X_t = x_t N_t$ and where N_t is population that grows at constant rate n, i.e., $\frac{N_{t+1}}{N_t} = 1 + n$.

- Note that we can rewrite this problem by dividing everything by N_t:

$$\max_{c_t \geq 0, k_{t+1} \geq 0, h_t \in [0,1]} \sum_{t=0}^{\infty} \beta^t \; u(c_t, h_t)$$

subject to the budget constraint,

$$c_t + k_{t+1}(1+n) = w_t \; h_t + (1 + r_t - \delta) \; k_t.$$

where we have used the fact that $\frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = k_{t+1}(1+n)$.

- Firms rent capital and solve a static problem:

$$\max_{h_t, K_t} \; Y_t - w_t h_t N_t - r_t K_t$$

subject to

$$Y_t = e^{z_t} F(K_t, h_t N_t)$$

and K_0 given.

- To make life easy, the model is already stationary.

Recursive competitive equilibrium (RCE)

A RCE is a set of functions $v(k, K, z)$, $c(k, K, z)$, $k'(k, K, z)$, $h'(k, K, z)$; prices $w(K, z)$ and $r(K, z)$; and an aggregate law of motion of capital $\Gamma_k(K, z)$ and aggregate labor $\Gamma_h(K, z)$ such that,[3]

1. Given w, r, Γ_K and Γ_h, the value function $v(k, K, z)$ solves the Bellman equation,

$$v(k, K, z) = \max_{c \geq 0, k' \geq 0, h \in [0,1]} u(c, h) + \beta E_{z'|z} v(k', K', z') \tag{1}$$

   such that

$$\begin{aligned} c + k' &= w(K, z)h + (1 + r(K, z) - \delta)k \\ z' &= \rho z + \varepsilon, \quad \varepsilon \sim N(0, \sigma_\varepsilon^2) \end{aligned}$$

2. Factor prices equate marginal products, $r(K, z) = e^z F_K(K, H)$ and $w(K, z) = e^z F_H(K, H)$.
3. Aggregate consistency, $K' = \Gamma(K, z) = k'(K, K, z)$ and $H = \Gamma(K, z) = h(K, K, z)$.
4. Market clearing, $e^z F(K, H) = c(K, K, z) + i(K, K, z)$.

[3] In general, note that the aggregate states of this economy are aggregate capital, K, and aggregate labor, H. However, given the CRS properties of the aggregate production function the capital-labor ratio will be a sufficient statistic to solve for this economy. This way, to ease the exposition, I avoid writing explicitly aggregate labor in the optimal functions.

Roadmap

1. Find the nonlinear system that characterizes the model economy.
2. Stationarize the economy.
3. Compute the steady state.
4. Locally approximate (1st order Taylor) the nonlinear system around the steady state. The system is linear in controls and state variables. [4] Some equations in the linear system are difference equations.
5. Express the linear system in some matrix representation. This representation is what differs by solution method. Then, use a matrix decomposition method to derive:
   - Optimal decision rules, i.e., linear functions from the state variables to the control variables.
   - Laws of motion for the endogenous state variables, i.e., linear functions from the state variables to the state variables in the next period.

[4] State variables can be exogenous or endogenous.

We assume that there is no population growth $n = 0$ and normalize population level to 1.

Let's assume parametric form for utility and production functions:

- Let's assume the utility function is,

$$u(c, h) = \ln c - \kappa \frac{h^{1+\frac{1}{\nu}}}{1 + \frac{1}{\nu}}$$

- and aggregate technology is,

$$e^z F(k, h) = e^z k^{1-\theta} h^{\theta}$$

Step 1: Characterize the solution (a nonlinear system)

- The solution is characterized by these set of equilibrium conditions:

$$
\begin{aligned}
\frac{1}{c} &= \beta \; E_{z'|z} \; \frac{1}{c'} \; (1 + r - \delta) \\
\frac{w}{c} &= \kappa \; h^{\frac{1}{\nu}} \\
r &= (1-\theta) e^{z} k^{-\theta} h^{\theta} \\
w &= \theta e^{z} k^{1-\theta} h^{\theta - 1} \\
c + k' &= wh + rk + (1-\delta) k \\
z' &= \rho z + \varepsilon
\end{aligned}
$$

  and the transversality condition.

- The equilibrium conditions of the model are the household FOC(k') [Euler Equation], household FOC(h), firm FOC(k), firm FOC(h), and the two aggregate laws of motion. Note that in the equilibrium conditions above we are already imposing aggregate consistency.

- There are other possibilities for the choice of variables (e.g. we could have used Lagrangian multipliers as variables).

Step 2: Stationarize the economy

- We have assumed above that there is no growth in TFP nor population. If any of these two happens, we will need to stationarize the economy either with a deterministc trend or a stochastic trend.

Step 3: Find the steady state

- Set the productivity shock to its unconditional mean, $z^* = z' = z = 0$.
- This implies

$$
\begin{aligned}
k^* = \quad & k' = \quad k \\
c^* = \quad & c' = \quad c \\
h^* = \quad & h' = \quad h \\
r^* = \quad & r' = \quad r \\
w^* = \quad & w' = \quad w
\end{aligned}
$$

- The steady-state equilibrium is

$$
\begin{aligned}
1 &= \beta(1 + r^* - \delta) \\
\frac{w^*}{c^*} &= \kappa \, {h^*}^{\frac{1}{\nu}} \\
r^* &= (1-\theta)e^{z^*} {k^*}^{-\theta} {h^*}^{\theta} \\
w^* &= \theta e^{z^*} {k^*}^{1-\theta} {h^*}^{\theta-1} \\
c^* &= w^* h^* + (r^* - \delta)k^* \\
z^* &= 0
\end{aligned}
$$

- Given values for $\frac{k^*}{y^*}$, $\frac{c^*}{y^*}$, h^*, θ, ν, and normalizing $y^* = 1$, we can calibrate the values of δ, β and κ.

Step 4: Locally approximate the nonlinear system around the steady state with a log-linear system

- Take each equation in the nonlinear system, and totally differentiate it around the steady state.

- A useful way to do so is by replacing each variable x by

$$x = x^* \, e^{\widehat{x}} \simeq x^*(1 + \widehat{x})$$

where $\widehat{x}$ is the log-deviation with respect to steady state, x^*. Note that $\widehat{x}\widehat{y} \simeq 0$.

- The nonlinear system is log-linearized as

$$
\begin{aligned}
\frac{1}{c^* e^{\widehat{c}}} &= \beta \; E_{z'|z} \; \frac{1}{c^* e^{\widehat{c'}}} \; (1 + r^* e^{\widehat{r'}} - \delta) \\
\frac{w^* e^{\widehat{w}}}{c^* e^{\widehat{c}}} &= \kappa \; \left(h^* e^{\widehat{h}} \right)^{\frac{1}{\nu}} \\
r^* e^{\widehat{r}} &= (1-\theta) e^{z} \left(k^* e^{\widehat{k}} \right)^{-\theta} \left(h^* e^{\widehat{h}} \right)^{\theta} \\
w^* e^{\widehat{w}} &= \theta e^{z} \left(k^* e^{\widehat{k}} \right)^{1-\theta} \left(h^* e^{\widehat{h}} \right)^{\theta-1} \\
c^* e^{\widehat{c}} + k^* e^{\widehat{k'}} &= w^* e^{\widehat{w}} h^* e^{\widehat{h}} + r^* e^{\widehat{r}} k^* e^{\widehat{k}} + (1-\delta) k^* e^{\widehat{k}} \\
z' &= \rho z + \varepsilon
\end{aligned}
$$

- Then, simplify using steady-state conditions ...

$$
\begin{aligned}
\frac{1}{e^{\widehat{c}}} &= E_{z'|z} \frac{1}{e^{\widehat{c'}}} \left(1 + \beta r^* \widehat{r'}\right) \\
\frac{e^{\widehat{w}}}{e^{\widehat{c}}} &= \left(e^{\widehat{h}}\right)^{\frac{1}{\nu}} \\
e^{\widehat{r}} &= e^{z} \left(e^{\widehat{k}}\right)^{-\theta} \left(e^{\widehat{h}}\right)^{\theta} \\
e^{\widehat{w}} &= e^{z} \left(e^{\widehat{k}}\right)^{1-\theta} \left(e^{\widehat{h}}\right)^{\theta-1} \\
c^*\widehat{c} + k^*\widehat{k'} &= w^* h^* (\widehat{w} + \widehat{h}) + r^* k^* (\widehat{r} + \widehat{k}) + (1-\delta) k^* \widehat{k} \\
z' &= \rho z + \varepsilon
\end{aligned}
$$

- **Remark**. The simplification of the Euler equation follows these steps:

$$\begin{aligned}
\frac{1}{c^* e^{\widehat{c}}} &= \beta\ E_{z'|z}\ \frac{1}{c^* e^{\widehat{c'}}}\ (1 + r^* e^{\widehat{r'}} - \delta) \\
\frac{1}{e^{\widehat{c}}} &= \beta\ E_{z'|z}\ \frac{1}{e^{\widehat{c'}}}\ (1 + r^*(1 + \widehat{r'}) - \delta) \\
\frac{1}{e^{\widehat{c}}} &= \beta\ E_{z'|z}\ \frac{1}{e^{\widehat{c'}}}\ (1 + r^* - \delta + r^* \widehat{r'}) \\
\frac{1}{e^{\widehat{c}}} &= E_{z'|z}\ \frac{1}{e^{\widehat{c'}}}\ (1 + \beta r^* \widehat{r'})
\end{aligned}$$

  and note that $\ln(1 + \beta r^* \widehat{r'}) \approx \beta r^* \widehat{r'}$.

- **Remark**. To simplify the budget constraint above we have used $e^{\widehat{x}} \simeq (1 + \widehat{x})$, and $\widehat{xy} \simeq 0$, as follows:

$$\begin{aligned}
c^* e^{\widehat{c}} + k^* e^{\widehat{k'}} &= w^* e^{\widehat{w}} h^* e^{\widehat{h}} + r^* e^{\widehat{r}} k^* e^{\widehat{k}} + (1-\delta) k^* e^{\widehat{k}} \\
c^*(1 + \widehat{c}) + k^*(1 + \widehat{k'}) &= w^*(1 + \widehat{w}) h^*(1 + \widehat{h}) + r^*(1 + \widehat{r}) k^*(1 + \widehat{k}) + (1-\delta) k^*(\\
c^* \widehat{c} + k^* \widehat{k'} &= w^* h^* (\widehat{w} + \widehat{h}) + r^* k^* (\widehat{r} + \widehat{k}) + (1-\delta) k^* \widehat{k}
\end{aligned}$$

- and taking natural logs (except for the productivity shock process, and the budget constraint)

$$
\begin{aligned}
-\widehat{c} &= E_{z'|z}[\beta r^* \widehat{r'} - \widehat{c'}] \\
\widehat{h} &= \nu(\widehat{w} - \widehat{c}) \\
\widehat{r} &= z + (-\theta)\widehat{k} + \theta\widehat{h} \\
\widehat{w} &= z + (1-\theta)\widehat{k} - (1-\theta)\widehat{h} \\
c^*\widehat{c} + k^*\widehat{k'} &= w^* h^*(\widehat{w} + \widehat{h}) + r^* k^*(\widehat{r} + \widehat{k}) + (1-\delta)k^*\widehat{k} \\
z' &= \rho z + \varepsilon
\end{aligned}
$$

- **Remark**. Linearization delivers a linear law of motion for the choice variables that displays *certainty equivalence*, i.e., it does not depend on σ.

- **Remark**. Note that here we have log-linearized the system. We could have instead linearized the system. Some practitioners have favored logs because the exact solution of the neoclassical growth model in the case of log utility and full depreciation is loglinear.

  This question is not completely settled (see a discussion in Aruoba, Fernández-Villaverde, and Rubio (2006))

Step 5: Matrix representation and decomposition of the linear system

Now, we express the linear system in some matrix representation to obtain the linearized decision rules using alternative matrix decomposition methods

- Eigenvalue decomposition, Blanchard and Kahn (1980)
- Undetermined coefficients, Uhlig (1997)

The method of undetermined coefficients deals with noninvertibility problems that may arise by the application of the Blanchard-Kahn method. QZ decomposition in Sims (2002), and the generalized Schur decomposition in Klein (2000), also resolve that problem.

The Blanchard and Kahn (1980) method: Eigenvalue decomposition

- Denote the number of exogenous state variables as n_z, the number of endogenous state variables as n_s, the total number of state variables $n = n_z + n_s$ and the number of control variables m.
- There is one exogenous state, z, that is, n_z=1.
- The number of endogenous state variables coincides with the number of equations that include an expectation operator, n_s=1.
- There is some degree of freedom in choosing the number of control variables, m. Our exercise includes four variables (c, h, w, r), that is, m=4. However, we can substitute out two equations (say w and r), leaving only c and h.

- Write the following matrix representation,

$$A_{(n+m)\times(n+m)} \left[\begin{array}{c} x'_{n\times 1} \\ Ey'_{m\times 1} \end{array} \right] = B_{(n+m)\times(n+m)} \left[\begin{array}{c} x \\ y \end{array} \right] + C_{(n+m)\times n_v} \; v'_{n_v\times 1}$$

  where $x = [z, k]'$ is the vector of state variables, and $y = [c, h, r, w]'$ is the vector of endogenous variables.

- The total number of equations is $n + m$, and E is an expectation operator with information at the current period.

- **Remark**. There is only one expecation operator, and it is associated with y'. That is, next period's state variable cannot be included in the expectation operator. If that were the case one has to substitute out the endogenous state variable by a control one.

- First, rearrange the linerized system:

$$
\begin{aligned}
z' &= \rho z + \varepsilon \\
E_{z'|z}[\beta r^* \widehat{r'} - \widehat{c'}] &= -\widehat{c} \\
0 &= \nu(\widehat{w} - \widehat{c}) - \widehat{h} \\
k^* \widehat{k'} &= w^* h^* (\widehat{w} + \widehat{h}) + r^* k^* (\widehat{r} + \widehat{k}) + (1-\delta) k^* \widehat{k} - c^* \widehat{c} + \\
0 &= z + (-\theta)\widehat{k} + \theta \widehat{h} - \widehat{r} \\
0 &= z + (1-\theta)\widehat{k} - (1-\theta)\widehat{h} - \widehat{w}
\end{aligned}
$$

- Explicitly, in our example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \beta r^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z' \\ k' \\ Ec' \\ Eh' \\ Er' \\ Ew' \end{bmatrix} =$$

$$\begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\nu & -1 & 0 & \nu \\ 0 & (1+r^*+\delta)k^* & -c^* & w^*h^* & r^*k^* & w^*h^* \\ 1 & -\theta & 0 & \theta & 1 & 0 \\ 1 & 1-\theta & 0 & -(1-\theta) & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ k \\ c \\ h \\ r \\ w \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon$$

- Now, multiply the system from the left A^{-1} (assuming that A is invertible).

- Then the matrix representation of the linear system becomes

$$\begin{bmatrix} x'_{n\times 1} \\ Ey'_{m\times 1} \end{bmatrix} = A^{-1}B \begin{bmatrix} x \\ y \end{bmatrix} + A^{-1}C\ v'$$

- **Remark**. In our example A is actually not invertible. Klein (2000) overcomes the potential noninvertibility of A by implementing a complex generalized Schur decomposition to decompose A and B. That is a generalization of the QZ decomposition that allows for complex eigenvalues associatied with A and B. The Schur decomposition of A and B are given by

$$\begin{aligned} Q\widetilde{A}Z &= S \\ Q\widetilde{B}Z &= R \end{aligned}$$

where (Q, Z) are unitary and (S, T) are upper triangular matrices with diagonal elements containing the generalized eigenvalues of $\widetilde{A}$ and $\widetilde{B}$.

- And simplifying notation,

$$\left[\begin{array}{c} x'_{n\times 1} \\ Ey'_{m\times 1} \end{array}\right] = F \left[\begin{array}{c} x \\ y \end{array}\right] + G\ v'$$

where $F = A^{-1}B$ is a $(n+m)\times(n+m)$ matrix, and G is a $(n+m)\times(n_v)$ matrix.

- Now, we apply the Jordan decomposition to the matrix F. [5] Recall that given a square matrix F, we want to choose a matrix M such that $M^{-1}FM$ is as nearly diagonal as possible. [6]

- The Jordan canonical form is

$$F = HJH^{-1} = [d_1, d_2, ..., d_{n+m}] \begin{bmatrix} \lambda_1 & 0 & 0 & . & 0 \\ 0 & \lambda_2 & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & \lambda_{n+m} \end{bmatrix} [d_1, d_2, ..., d_{n+m}]^{-1}$$

where $\{\lambda_i\}_{i=1}^{n+m}$ are eigenvalues and the column vectors $\{d_i\}_{i=1}^{n+m}$ are associated eigenvectors.

[5] See, for instance, appendix B in Gilbert and Strang.

[6] In the simplest case, F has a complete set of eigenvectos and they become the columns of M.

- The eigenvalues $\{\lambda_i\}_{i=1}^{n+m}$ are ordered such that

$$|\lambda_1| < |\lambda_2| < ... < |\lambda_{n+m}|.$$

- Let the number of eigenvalues outside the unit circle be h.

- **Blanchard-Kahn Conditions**

  1. If $h = m$, the solution to the system is unique (saddle path stable)
  2. If $h > m$, theres is no solution to the system.
  3. If $h < m$, there are infinite solutions (indeterminacy).

- Assume we have a unique solution, i.e., $h = m$.

- Let's partition the matrix J such that the upper-left block contains only the eigenvalues inside the unit circle. Partition the matrix G accordingly.

- Then,

$$\begin{bmatrix} x' \\ Ey' \end{bmatrix} = H \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} H^{-1} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} v'$$

- Note that J_2 is said to be explosive because J_2^n diverges to infinite as n increases.

- Multiply H^{-1} from the left and we get:

- Then,

$$H^{-1} \begin{bmatrix} x' \\ Ey' \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} H^{-1} \begin{bmatrix} x \\ y \end{bmatrix} + H^{-1} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} v'$$

- Partition H and H^{-1} as follows:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

where H_{11} is conformable with J_1, and

$$H^{-1} = \begin{bmatrix} \widehat{H}_{11} & \widehat{H}_{12} \\ \widehat{H}_{21} & \widehat{H}_{22} \end{bmatrix}$$

- Then, use

$$\begin{bmatrix} \widetilde{x} \\ \widetilde{y} \end{bmatrix} = \begin{bmatrix} \widehat{H}_{11} & \widehat{H}_{12} \\ \widehat{H}_{21} & \widehat{H}_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

to rearrange

$$\begin{bmatrix} \widetilde{x}' \\ E\widetilde{y}' \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ \widetilde{y} \end{bmatrix} + \begin{bmatrix} \widetilde{G}_1 \\ \widetilde{G}_2 \end{bmatrix} v'$$

- Recall that J is diagonal, so the upper part and the lower part can be easily separated. We aim at transforming the system so that control variables depend upon only the unstable eigenvalues of A contained in J_2.

- The unstable piece of the system is

$$E\widetilde{y}' = J_2\widetilde{y} + \widetilde{G}_2\ v'$$

- Now, isolate

$$\widetilde{y} = J_2^{-1} E \widetilde{y}' - J_2^{-1} \widetilde{G}_2 \; v' \tag{2}$$

and forward one period

$$\widetilde{y}' = J_2^{-1} E \widetilde{y}'' - J_2^{-1} \widetilde{G}_2 \; v'' \tag{3}$$

- and use the law of iterated expecations $E_t E_{t+1}(x_t) = E_t(x_t)$ to rewrite (3) as:

$$E\widetilde{y}' = J_2^{-1} E \widetilde{y}'' - J_2^{-1} \widetilde{G}_2 \; Ev''$$

that we can plug back into (2):

$$\widetilde{y} = J_2^{-2} E \widetilde{y}'' - J_2^{-2} \widetilde{G}_2 \; Ev'' - J_2^{-1} \widetilde{G}_2 \; v'$$

- If we keep iterating forward we get

$$\widetilde{y} \quad = \quad -J_2^{-1}\widetilde{G}_2\ v' - J_2^{-2}\widetilde{G}_2\ Ev'' - J_2^{-2}\widetilde{G}_3\ Ev'''... \qquad (4)$$

- Then (4) can be simplified as

$$\widetilde{y} = -J_2^{-1}\widetilde{G}_2\ v'$$

where we have used $E(v') = E(v'') = E(v''') = ... = 0$ and $\lim_{n\to\infty} J_2^{-n} = 0$ because J_2 contains the explosive eigenvalues.

- Plugging back the formula for $\widetilde{y}$,

$$\begin{aligned}\widetilde{y} &= -J^{-1}\widetilde{G}_2 v' \\ \left[\widehat{H}_{21} + \widehat{H}_{22}\right]\ \widetilde{y} &= -J^{-1}\ \left[\widehat{H}_{21}\ G_1 + \widehat{H}_{22}\ G_2\right]\ v'\end{aligned}$$

- That is, the optimal decision rules (i.e., mappings from x and v' to y) are

$$\widetilde{y} = -\widehat{H}_{22}^{-1}\ \widehat{H}_{21}\ x - \widehat{H}_{22}^{-1}\ J_2^{-1}\ \left[\widehat{H}_{21}\ G_1 + \widehat{H}_{22}\ G_2\right]$$

- Once we have the decision rules, we can use the law of motion to obtain x'.

The method of undetermined coefficients, Uhligh (1997)

- We deal with the same linearized model as in the previous section:

$$
\begin{aligned}
0 &= E_{z'|z}[\beta r^* \widehat{r'} - \widehat{c'} + \widehat{c}] \\
\widehat{h} &= \nu(\widehat{w} - \widehat{c}) \\
\widehat{r} &= z + (-\theta)\widehat{k} + \theta\widehat{h} \\
\widehat{w} &= z + (1-\theta)\widehat{k} - (1-\theta)\widehat{h} \\
c^*\widehat{c} + k^*\widehat{k'} &= w^* h^*(\widehat{w} + \widehat{h}) + r^* k^*(\widehat{r} + \widehat{k}) + (1-\delta)k^*\widehat{k} \\
z' &= \rho z + \varepsilon' \\
E[\varepsilon'] &= 0
\end{aligned}
$$

- Substitute away factor prices,

$$
\begin{aligned}
0 &= E_{z'|z}[\beta r^*(z' + (-\theta)\widehat{k'} + \theta\widehat{h'}) - \widehat{c'} + \widehat{c}] &(5)\\
\widehat{h} &= \nu(z + (1-\theta)\widehat{k} - (1-\theta)\widehat{h} - \widehat{c}) &(6)\\
c^*\widehat{c} + k^*\widehat{k'} &= y^*\widehat{y} + (1-\delta)k^*\widehat{k} &(7)\\
z' &= \rho z + \varepsilon &(8)\\
E[\varepsilon'] &= 0 &(9)
\end{aligned}
$$

where to simplify notation we are using the Euler theorem:

$$
\begin{aligned}
y^*\widehat{y} &= w^*h^*(z + (1-\theta)\widehat{k} - (1-\theta)\widehat{h} + \widehat{h}) + r^*k^*(z + (-\theta)\widehat{k} + \theta\widehat{h} + \widehat{k})\\
&= y^*(z + (1-\theta)\widehat{k} + \theta\widehat{h})
\end{aligned}
$$

- Substitute away consumption. To do so, isolate consumption from the FOC of labor (6),

$$\widehat{c} \quad = \quad z + (1-\theta)\widehat{k} - (1-\theta)\widehat{h} - \frac{1}{\nu}\widehat{h}$$

- Plug consumption into the Euler equation (5)

$$\begin{aligned} 0 \quad = \quad & E_{z'|z}[\beta r^*(z' + (-\theta)\widehat{k'} + \theta\widehat{h'}) - (z' + (1-\theta)\widehat{k'} - (1-\theta)\widehat{h'} - \frac{1}{\nu}\widehat{h'})] \\ & + z + (1-\theta)\widehat{k} - (1-\theta)\widehat{h} - \frac{1}{\nu}\widehat{h} \end{aligned}$$

and the budget constraint (7):

$$c^*(z + (1-\theta)\widehat{k} - (1-\theta)\widehat{h} - \frac{1}{\nu}\widehat{h}) + k^*\widehat{k'} \quad = \quad y^*(z + (1-\theta)\widehat{k} + \theta\widehat{h}) + (1-\delta)k^*\widehat{k}$$

- That is, the system (5)-(9) reduces to

$$0 = E\psi_1 \widehat{k}' + \psi_2 \widehat{k} + \psi_3 \widehat{h}' + \psi_4 \widehat{h} + \psi_5 z' + \psi_6 z \quad (10)$$

$$0 = \eta_1 \widehat{k}' + \eta_2 \widehat{k} + \eta_3 \widehat{h} + \eta_4 z \quad (11)$$

$$z' = \rho\, z + \varepsilon' \quad (12)$$

- Where

$$
\begin{aligned}
\widehat{k'}: \quad \psi_1 &= \beta r^*(-\theta) - (1-\theta) \\
\widehat{k}: \quad \psi_2 &= -(1-\theta) \\
\widehat{h'}: \quad \psi_3 &= \beta r^*\theta - ((\theta-1) - \frac{1}{\nu}) \\
\widehat{h}: \quad \psi_4 &= -((\theta-1) - \frac{1}{\nu}) \\
\widehat{z'}: \quad \psi_5 &= \beta r^* - 1 \\
\widehat{z}: \quad \psi_6 &= -1
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{k'}: \quad \eta_1 &= -k^* \\
\widehat{k}: \quad \eta_2 &= -c^*(1-\theta) + y^*(1-\theta) + (1-\delta)k^* \\
&= i^*(1-\theta) + (1-\delta)k^* \\
&= \delta k^*(1-\theta) + (1-\delta)k^* \\
&= (1+\delta\theta)k^* \\
\widehat{h}: \quad \eta_3 &= -c^*((\theta-1) - \frac{1}{\nu}) + y^*\theta \\
\widehat{z}: \quad \eta_4 &= -c^* + y^* = \delta k^*
\end{aligned}
$$

- Postulate the decision rules for $\widehat{k}'$ and $\widehat{h}$ to be

$$\widehat{k}' = \phi_1 \widehat{k} + \phi_2 z \tag{13}$$

$$\widehat{h} = \phi_3 \widehat{k} + \phi_4 z \tag{14}$$

- That is, we postulate decision rules that are linear in $\widehat{k}$ and $\widehat{z}$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are unknowns.

- We also know that

$$z' = \rho z + \varepsilon'. \tag{15}$$

- Plug (13)-(15) into the system (10)-(12) to obtain:

$$\begin{aligned}
0 &= E\psi_1\widehat{k}' + \psi_2\widehat{k} + \psi_3\widehat{h}' + \psi_4\widehat{h} + \psi_5 z' + \psi_6 z \\
&= E\psi_1(\phi_1\widehat{k} + \phi_2 z) + \psi_2\widehat{k} + \psi_3(\phi_3\widehat{k}' + \phi_4 z') + \psi_4(\phi_3\widehat{k} + \phi_4 z) + \psi_5(\rho\ z + \varepsilon') + \psi_6 z \\
&= E\psi_1(\phi_1\widehat{k} + \phi_2 z) + \psi_2\widehat{k} + \psi_3(\phi_3(\phi_1\widehat{k} + \phi_2 z) + \phi_4(\rho\ z + \varepsilon')) \\
&\qquad + \psi_4(\phi_3\widehat{k} + \phi_4 z) + \psi_5(\rho\ z + \varepsilon') + \psi_6 z \\
&= E(\psi_1\phi_1 + \psi_2 + \psi_3\phi_3\phi_1 + \psi_4\phi_3)\ \widehat{k} + (\psi_1\phi_2 + \psi_3\phi_3\phi_2 + \psi_3\phi_4\rho + \psi_4\phi_4 + \psi_5\rho + \psi_6)\ z
\end{aligned}$$

and

$$\begin{aligned}
0 &= \eta_1\widehat{k}' + \eta_2\widehat{k} + \eta_3\widehat{h} + \eta_4 z \\
&= \eta_1(\phi_1\widehat{k} + \phi_2 z) + \eta_2\widehat{k} + \eta_3(\phi_3\widehat{k} + \phi_4 z) + \eta_4 z \\
&= (\eta_1\phi_1 + \eta_2 + \eta_3\phi_3)\widehat{k} + (\eta_1\phi_2 + \eta_3\phi_4 + \eta_4)z
\end{aligned}$$

where we have used $E(\varepsilon') = 0$.

- That is...

- That is,

$$0 = \gamma_1 \widehat{k} + \gamma_2 z \quad (16)$$
$$0 = \gamma_3 \widehat{k} + \gamma_4 z \quad (17)$$

where,

$$\gamma_1 = \psi_1\phi_1 + \psi_2 + \psi_3\phi_3\phi_1 + \psi_4\phi_3$$
$$\gamma_2 = \psi_1\phi_2 + \psi_3\phi_3\phi_2 + \psi_3\phi_4\rho + \psi_4\phi_4 + \psi_5\rho + \psi_6$$
$$\gamma_3 = \eta_1\phi_1 + \eta_2 + \eta_3\phi_3$$
$$\gamma_4 = \eta_1\phi_2 + \eta_3\phi_4 + \eta_4$$

- Then, there are four unknowns in (16) and (17), $\{\phi_1, \phi_2, \phi_3, \phi_4\}$.

- To solve for the uknonwns, set $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = 0$

- General solution method for undetermined coefficent:

$$
\begin{aligned}
0 &= Ax' + Bx + Cy + Dz \\
0 &= EFx'' + Gx' + Hx + Jy' + Ky + Lz' + Mz \\
z' &= Nz + \varepsilon' \quad \text{and} \quad E(\varepsilon') = 0.
\end{aligned}
$$

where N has only stable eigenvalues. x are $(m \times 1)$ endogenous state variables, z are $(k \times 1)$ exogenous state variables, and y are $(n \times 1)$ control variables, and ε' is a vector of shocks $(k \times 1)$.

- Then guess the optimal decision rule to be

$$\begin{aligned} x' &= Px + Qz \\ y &= Rx + Sz \end{aligned}$$

where P, Q, R and S are unknowns.

Investment shocks: Greenwood, Hercowitz, and Krusell (1997,2000) and Fisher (2006)

In the case of the one sector formulation we replace:

- Production function:
$$y_t = e^{a_t} F(k_t, h_t)$$
- Investment equation:
$$e^{v_t} i_t = k_{t+1} - (1-\delta)k_t$$
where $V_t = V_0(1+\lambda_v)e^{v_t}$ is investment-specific technical change and we assume $V_0 = 1$ and $\lambda_v = 0$.

Capital utilization: King and Rebelo (1999) and Christiano, Eichenbaum and Evans (2005)

- Production function:

$$y_t = e^{a_t} F(u_t k_t, h_t)$$

- Investment equation:

$$e^{v_t} i_t = k_{t+1} - (1 - \delta(u_t))k_t$$

where $\delta(u_t) = \delta_0 + \delta_1 \left(u_t^{1+\frac{1}{\xi}} - 1 \right)$ where ξ is the elasticity of depreciation.

Habit persistence: Boldrin, Christiano, and Fisher (2001)

- Utility function:

$$u(c_t, h_t) = \ln\left(c_t - \eta c_{t-1}\right) - \kappa \frac{h_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}$$

where η is a consumption habit parameter.

The intensive and extensive margin: Hansen (1985) and Cho and Cooley (1994)

We model the intensive h_t hours per day, and extensive e_t days per year, separately. The extensive margin can have different interpretations. In our example, the individual extensive margin is days worked per year and the aggregate extensive margin is the employtment rate, e_t.[7]

- Utility function:

$$u(c_t, h_t, e_t) = g(c) - v(h)e - \phi(e)e$$

where

$$g(c_t) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad v(h_t) = \kappa_h \frac{h_t^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}}, \qquad \phi(e_t) = \kappa_e \frac{e_t^{\frac{1}{\nu_e}}}{1+\frac{1}{\nu_e}}$$

[7] That is, e_t is the aggregate number of days worked by all individuals over the aggregate number of days available to all individuals in the labor force.

- That is,

$$\max_{c_t \geq 0, h_t \in (0,1), e_t \in (0,1)} \sum_t^{\infty} \beta^t u(c_t, h_t, e_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \kappa_h \frac{h_t^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}} e_t - \kappa_e \frac{e_t^{1+\frac{1}{\nu_e}}}{1+\frac{1}{\nu_e}}$$

subject to

$$c_t + k_{t+1}(1+n) = w_t h_t e_t + (1 + r_t - \delta) k_t$$

- Then, FOC(h) is

$$c^{-\sigma} we = \kappa_h h^{\frac{1}{\nu_h}} e$$
$$c^{-\sigma} w = \kappa_h h^{\frac{1}{\nu_h}}$$
$$h = \left(\frac{1}{\kappa_h}\right)^{\nu_h} \left(c^{-\sigma} w\right)^{\nu_h}.$$

Hence, holding everything else constant,

$$\frac{\partial h}{\partial w} = \nu_h \left(\frac{1}{\kappa_h}\right)^{\nu_h} \left(c^{-\sigma} w\right)^{\nu_h - 1} c^{-\sigma} = \nu_h \left(\frac{1}{\kappa_h}\right)^{\nu_h} \left(c^{-\sigma} w\right)^{\nu_h} \frac{c^{-\sigma}}{c^{-\sigma} w} = \nu_h \frac{h}{w}$$

That is, the elasticity of the intensive labor supply h_t with respect to wages is ν_h,

$$\frac{\partial h}{\partial w} \frac{w}{h} = \nu_h.$$

- Then, FOC(e) is

$$c^{-\sigma} wh - \kappa_h \frac{h_t^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}} = \kappa_e e^{\frac{1}{\nu_e}}$$

$$\kappa_h h^{\frac{1}{\nu_h}} h - \kappa_h \frac{h_t^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}} = \kappa_e e^{\frac{1}{\nu_e}}$$

$$\kappa_h h^{1+\frac{1}{\nu_h}} - \kappa_h \frac{h_t^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}} = \kappa_e e^{\frac{1}{\nu_e}}$$

$$\left(1+\frac{1}{\nu_h}\right) \kappa_h \frac{h^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}} - \kappa_h \frac{h_t^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}} = \kappa_e e^{\frac{1}{\nu_e}}$$

$$\frac{1}{\nu_h} \kappa_h \frac{h^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}} = \kappa_e e^{\frac{1}{\nu_e}}$$

$$\frac{1}{\nu_h} v(h) = \kappa_e e^{\frac{1}{\nu_e}}$$

$$e = \left(\frac{1}{\kappa_e}\right)^{\nu_e} \left(\frac{1}{\nu_h} v(h)\right)^{\nu_e}$$

where we have used the fact that, from FOC(h_t), $c^{-\sigma} w = \kappa_h h^{\frac{1}{\nu_h}}$.

Hence, holding everything else constant,

$$
\begin{aligned}
\frac{\partial e}{\partial w} &= \nu_e \left(\frac{1}{\kappa_e}\right)^{\nu_e} \left(\frac{1}{\nu_h} v(h)\right)^{\nu_e - 1} \frac{1}{\nu_h} \frac{\partial v(h)}{\partial h} \frac{\partial h}{\partial w} \\
&= \nu_e \left(\frac{1}{\kappa_e}\right)^{\nu_e} \left(\frac{1}{\nu_h} v(h)\right)^{\nu_e - 1} \frac{1}{\nu_h} \kappa_h h^{\frac{1}{\nu_h}} \frac{\partial h}{\partial w} \\
&= \nu_e \left(\frac{1}{\kappa_e}\right)^{\nu_e} \left(\frac{1}{\nu_h} v(h)\right)^{\nu_e - 1} \frac{1}{\nu_h} \kappa_h h^{\frac{1}{\nu_h}} \nu_h \frac{h}{w} \\
&= \nu_e e \frac{1}{\frac{1}{\nu_h} v(h)} \frac{1}{\nu_h} \kappa_h h^{\frac{1}{\nu_h}} \nu_h \frac{h}{w} = \nu_e e \frac{1}{\frac{1}{\nu_h} v(h)} \kappa_h h^{\frac{1}{\nu_h}} \frac{h}{w} = \nu_e e \frac{1}{\frac{1}{\nu_h} v(h)} \kappa_h h^{1+\frac{1}{\nu_h}} \frac{1}{w} \\
&= \nu_e e \frac{1}{\frac{1}{\nu_h} \kappa_h \frac{h_t^{1+\frac{1}{\nu_h}}}{1+\frac{1}{\nu_h}}} \kappa_h h^{1+\frac{1}{\nu_h}} \frac{1}{w} = \nu_e e \frac{1}{\frac{1}{\nu_h} \frac{1}{1+\frac{1}{\nu_h}}} \frac{1}{w} \\
&= \nu_e (1 + \nu_h) \frac{e}{w}.
\end{aligned}
$$

That is, the elasticity of the extensive labor supply e_t with respect to wages is $\nu_e(1 + \nu_h)$,

$$
\frac{\partial e}{\partial w} \frac{w}{e} = \nu_e(1 + \nu_h).
$$

- Then, the elasticity of labor supply $h_t e_t$ with respect to wages,

$$
\begin{aligned}
\frac{\partial he}{\partial w}\frac{w}{he} &= \left(\frac{\partial h}{\partial w}e + h\frac{\partial e}{\partial w}\right)\frac{w}{he} \\
&= \frac{\partial h}{\partial w}\frac{w}{h} + \frac{\partial e}{\partial w}\frac{w}{e} \\
&= \nu_h + \nu_e(1+\nu_h).
\end{aligned}
$$

- Then, FOC(k_{t+1}) is

$$\beta^t c_t^{-\sigma}(1+n)(-1) + \beta^{t+1} c_{t+1}^{-\sigma}(1 + r_{t+1} - \delta) = 0$$
$$c_t^{-\sigma}(1+n) = \beta c_{t+1}^{-\sigma}(1 + r_{t+1} - \delta),$$

that is,

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta \frac{N_t}{N_{t+1}}(1 + r_{t+1} - \delta),$$

Taking stock,

- FOC(h) is

$$h_t = \left(\frac{1}{\kappa_h}\right)^{\nu_h} \left(c_t^{-\sigma} w_t\right)^{\nu_h}$$

- FOC(e_t) is

$$e_t = \left(\frac{1}{\kappa_e}\right)^{\nu_e} \left(\frac{1}{\nu_h + 1} \kappa_h h_t^{1+\frac{1}{\nu_h}}\right)^{\nu_e}$$

- FOC(k_{t+1}) is

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta \frac{N_t}{N_{t+1}} (1 + r_{t+1} - \delta)$$

- Factor prices (w_t) is

$$w_t = e^{z_t} \left(\frac{k}{eh}\right)^{1-\theta}$$

- Factor prices (r_t) is

$$r_t = e^{z_t} \left(\frac{eh}{k}\right)^{\theta}$$